

## Chapter 18

# Effects of Gravitation

### 18.1 Curvature around a Massive Body

### 18.2 The Universe

#### 18.2.1 Background Ideas

After 1916, Einstein and others applied the General Theory of Relativity, the modern theory of gravity to the entire universe. This is the subject called cosmology, the study of the physics of the universe. There were earlier attempts to apply the physics known at the time to clarify observations of the world at large but the observational information was rather scant, structures such as galaxies had not yet been seen, and the observations that lead to the idea of the expanding universe had not yet been made, see Section 18.2.4 on page 380. In addition, the conclusions using Newton's theory of gravity suffered from technical difficulties, see Section 18.2.6 on page 389. In that section, we discuss a simple expanding Newtonian expansion since the physics is closer to our experience helps our understanding of the mechanisms of gravitation in determining the nature of the large scale universe. One thing was clear from the beginning. The force system that set the properties of the large scale structure of the universe was gravity. The electromagnetic force was negligible on large scales since it was so strong that matter in the large had to be electrically neutral and thus all the residual electric and magnetic fields were small<sup>1</sup>.

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<sup>1</sup>If there were pockets of one type of charged matter, the repulsion of the like charges would cause it to spread rapidly and dissipate. If there were separated pockets of differently charged matter, there would be significant recombination radiation at the interfaces.

The basic ideas of the nature of the large scale structure are so simple and compelling that it seems that they must be correct and, fortunately, most of the observational data are in complete concordance. Despite this simplicity, the history of the subject is full of surprising turns and it is worthwhile telling some of this history so that we can understand the context of our current understanding and why this is still an exciting and active research field – hardly a week goes by without some new article in the newspapers indicating some controversial measurement. Like all good science, cosmology is now being driven by new experimental results.

Also, It is important to realize that some of the current controversies in our understanding of the operation of the universe are all really at the interface of General Relativity and micro-physics. In this section, we will deal only with the broadly accepted aspects of the subject and leave the issues that emerge from the interaction of the large scale universe with micro-physics to a later chapter, see Chapter 23 on page 491. Because of this, in this chapter, we will treat the matter in universe very simply and accept forms of matter that are currently not completely understood.

Einstein had a rather simple outlook on the nature of the universe and its origin. Like Descartes and others before him, he felt that that the universe has always been present or at least reasonably stable. This desire was tempered though by the observation that, although the ages of the sun and planets were quite large, there were certainly dynamical processes taking place in the cosmos. This balance between perpetuity and evolution meant that he wanted solutions for the space-time structure of the universe that had stationary or at least quasi-stationary solutions, i. e. solutions that were stable over long periods of time. We should realize that the astronomy of the period was not nearly as advanced as it is today and the observational situation was that, at all distances, the night sky looked the same. Due to the fact that the speed of light is finite, looking at longer distances was the same as looking back in time. It is just that the distances that we being observed were small compared to what we now know are relevant to cosmological questions. Also, we have been observing the universe seriously for only the last few hundred years and on the lifetime of stars and things like that this is but an instant.

As they were originally proposed the equations for the evolution of space-time, the Einstein equations ?? on page ??, did not possess any stationary solutions; there were not enough dimensional parameters to define a time. He realized that there was a simple way to modify the equations and he

added the term now called the cosmological constant.

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} - \Lambda g^{\mu\nu} = -8\pi GT^{\mu\nu} \quad (18.1)$$

where  $\Lambda$  is the cosmological constant. With this term added, he was able to construct solutions that were stable over long times. Note that the cosmological constant has the same dimensions as the curvature,  $R$ , which is an inverse length squared. The equations now have three fundamental dimensional constants,  $c$ ,  $G$ , and  $\Lambda$ .

Two things changed the situation. The great astronomer Hubble observed that the distant galaxies were receding and the the rate of recession was proportional to the distance. We will discuss this observation in more detail in Section 18.2.4 on page 380 This observation freed Einstein from the illusion that the universe had to be stationary and he dropped the added terms. In 1922, Avner Freedman produced a set of solutions for the structure of space-time for the universe without the use of the cosmological constant, that were very compelling. In a sense, the Hubble observation allowed Einstein to accept the Freedman solutions as a basis for studies of the structure of the universe. There was another reason that it was easy to accept an expanding universe. Olber predicted that in a stationary universe the night sky should be bright, Section 18.2.3 on page 378. It is not. Thus with the availability of the Freedman solutions of his equations without the cosmological constant, Einstein dropped the cosmological constant term from his equations and considered his addition of it to them “his greatest mistake” since they were not a natural geometric element.

From the beginning, the Freedman model of the universe was ambiguous about some of the important features of the universe such as its general geometry. Observational data was not only insufficient to resolve these questions, it was also ambiguous. The primary issue centered on whether or not the expansion was slowing down. The acceleration of the universe is hard to observe directly. We have been observing the universe seriously for only a small fraction of its lifetime. The nature of the acceleration of the universe is determined by the energy/matter terms in the Einstein Equation, Equation ?? on page ?. The density of matter in the univers eis also difficult to measure and what measurements were available were not consistent with the dynamics of galaxies and clusters of galaxies, see Section 18.2.6 on page 399. Again, through the Einstein Equations, whether or not the expansion was slowing down or speeding up was connected to the question of whether the average curvature was positive or negative. Neither question could be answered.

As would be expected, the Freedman-Hubble expanding universe was not the only candidate for a model of the universe but its theoretical basis was so compelling that it was widely accepted. Not only was the average energy/matter density of the universe important for acceleration, its makeup was determined by the early thermal history of the universe. Speculation on the nature of the matter in the universe was, of course, determined by the micro-physics of the period. Although the nature of the cosmic distribution of matter was difficult to determine observationally, it was clear early on that the matter in the universe was dominantly light nuclei, electrons, and photons. Using information about this mix, it became possible to assign a temperature to the universe. The observation of the  $3^{\circ}\text{K}$  background radiation in 1964 by Penzias and Wilson which was predicted within the context of a hot Freedman-Hubble model confirmed this family of models but, because it also contained this requirement that the universe be hot, also led to the acceptance of the unfortunate name – Big Bang Cosmology, see Section 18.2.8 on page 401.

Despite its great success, Big Bang Cosmology had several disturbing features, see Section 18.2.9 on page 401. Although it is natural to expect that the universe was reasonably homogeneous initially, the observational data was simply too good. There were also predictions from micro-physics of particle species that should have been formed in the early universe and are not observed. Not surprisingly, advances in micro-physics now called the Standard Model, see Section 18.2.11 on page 401, implied a mechanism for the initiation of the expansion. This is the Inflationary Theory of the early universe, see Section 23.2 on page 491. More recently, a great deal of observational data of large scale systems has produced more questions and even reopened the question of the role of the cosmological constant, see Section 18.2.10 on page 401. In fact, the experimental situation with the large scale features of the universe is so compelling that many people are turning to questioning our understanding of the micro-physics that we are using. This is an exciting time to be dealing with cosmological physics. There has emerged a Standard Model of Cosmology that has such a secure observational basis that it is now a seriously accepted challenge to the simple Big Bang Theory.

### 18.2.2 Copernican Principle

It got Galileo into a great deal of trouble with the church but, today, we have no trouble convincing anyone that the earth is not the center of the universe. Not only that, we have no trouble convincing most people that

the sun is also not the center of the universe. You can also get people to accept the idea that the universe is homogeneous, i. e. the laws of physics are the same everywhere or, said another way, the universe is symmetric under space translations. Despite the acceptance of this statement, it is still difficult to convince people that there is no center and no boundary to the universe. The lack of a center or a boundary is an immediate consequence of homogeneity.

It is a fundamental assertion of cosmology that the universe is homogeneous and isotropic. Homogeneous means that there is no experiment that can be carried out that can distinguish one place from any other. This is not to say that there are not special circumstances that exist at certain places; we living on the earth are in a place that is complex, fruitful, and supports life. This is considered a special consequence of the history of this place and an accidental small fluctuation in the vast emptiness of the universe. Similarly, isotropic means that at any point all directions are the same. It is easy to think of spaces that are homogeneous and not isotropic, a cylinder for instance. Regardless, if all points are the same, there can be no point being distinguished as a center or a point on an edge. Regardless, the idea of a homogeneous universe has to be tested experimentally. The idea that the same laws of physics hold everywhere is well tested. Galaxy counts and the background glow of the universe agree with the assertion of homogeneity and isotropy to within expected limits, see Section 18.2.3 on page 378.

In a very real sense, this name of Big Bang does not help. Most explosions have a center and certainly all have an edge. This is contrary to our expectations for the universe. The length scale for which the homogeneity holds is one in which the galaxy is a point and even the fact that we are in a small cluster of galaxies is a local density fluctuation that, although intergalactic, is still on a small scale.

The homogeneity assumption implies that the important physical variables, such as density and so forth, must be independent of position. As stated above, it also means that the laws of physics hold at all places. This is probably the best general test of homogeneity. At large distances, stars work the same way as they do in our galaxy. In addition, all deep sky surveys are consistent with homogeneity at the largest distances. Otherwise, it is hard to make a direct test of homogeneity since we have only occupied this small piece of the cosmos. We are in an awkward situation. We are trying to construct a theory of the universe and we have little experience in it both spatially and temporally.

Isotropy is the statement that at any point all directions are the same. Again, there is no experiment that can differentiate one direction from an-

other. Here we can at least test this hypothesis locally by examining phenomena in all directions. The strongest test of isotropy is the  $3^0\text{K}$  background radiation, see Section 18.2.8 on page 401, which can be tested in all directions. Other than expected small fluctuations, it is shockingly isotropic, maybe too much so, see Section 18.2.9 on page 401.

Another important test of the homogeneity and isotropy assumption is the pattern of the Hubble Expansion, see Section 18.2.4 on page 380. The requirement of homogeneity and isotropy restricts the form of the relationship between velocity and distance for remote systems that was observed by Hubble. In fact, the assumptions of homogeneity and isotropy can be used to predict its form uniquely. The fact that it is consistent observationally is verification of these principles.

### 18.2.3 Olber's Paradox

#### The Paradox

This was one of the earliest indications that a permanent unchanging universe was not tenable. Basically it is the observation that, in a homogeneous steady universe, the night sky should be bright. Since it is not, there is a problem.

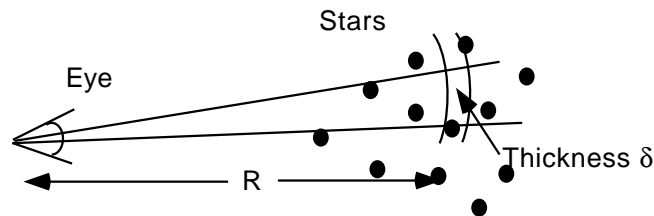


Figure 18.1: **Olber's Paradox** The number of stars that are in a shell of thickness  $\delta$  in the field of vision at a distance  $R$  is proportional to the distance squared. The brightness from each star at the eye falls off as  $R^{-2}$ . In a homogeneous universe, the density of stars is the same everywhere and the brightness is the same. Thus the brightness received at the eye is independent of distance and thus the sky should be bright.

The basis of this prediction is that as you look out at the night sky, since you see into some finite opening angle, the number of stars that are in your field of view from some distance  $R$ , grows as  $R^2$ , see Figure 18.1 on page 378. At the same time, the brightness of the light from a star at a distance  $R$  falls off with distance as  $R^{-2}$ . Therefore, in a homogeneous steady universe, the

net light, the number of stars times the brightness per star, received from the stars is independent of the distance. Adding up the contributions from all distances leads to a very large intensity, a bright night sky. Another way to look at it is to realize that in a homogeneous infinite universe along any direction your sight line must ultimately hit a star. This is Olber's Paradox – why is the night sky dark?

Of course, this picture has to be modified in modern times by our realization that the stars that we see are residents in our local galaxy and that, on the large scale, the points of light in the sky are identified not with stars but with galaxies. Substitute the word galaxy for star in the above explanation and you have the modern version of Olber's paradox.

The earlier explanation was that since there was dust or gases in the cosmos, the light falls off faster than  $R^{-2}$  and thus we see the dark patches caused by the extra absorption from the intervening material. In a unchanging universe, this explanation will not work. The intervening dust would absorb the light and heat up and glow until its glow balanced the light being absorbed, see Section ?? on page ?. Thus, if the universe is infinite and forever, there should not be a dark night sky.

Although our original interpretation of the dark spaces that are between the bright spots are nothing, we will find out that they are a very interesting and rich structure that providing us with a great deal of information about the early universe. Remember that since the speed of light is finite that seeing further is seeing earlier. With this in mind, we now can ask this dark is and begin to understand the way it came about.

### The Modern Resolution

We will get ahead of our story but it is good to understand the modern resolution of Olber's Paradox. The resolution of the paradox is that the universe is expanding and dynamic, see Section 18.2.4 on page 380. When looking out, we are really looking back in time and, at these earlier times, the stars and galaxies have not yet formed. Thus looking out and back in time in most directions we are seeing between the stars and galaxies; directions in which in the universe has no glowing object has yet formed. This is a place that has only the widely scattered matter and radiant energy and by definition is not glowing. The issue is that, if the universe is expanding and we are looking back, this portion of the universe that was once very dense. Not only very dense but also very hot. In the our model, this is light from a hot dense homogeneous aggregation of matter and radiation. In fact, what we see in the interval between the galaxies is the light from the universe

when it was about 300,000 years old. At this time, the universe was a hot sea of matter and mostly photons. The light that comes into our detectors now is the light of last scatter off the surface of this hot body. We cannot see any earlier because that light does not stream out. This is very similar to what we see coming from the sun. Sun light is the light from the outer surface. The interior of the sun is much hotter than the surface but we see light only from the outer most layer which is the surface of last scattering for the light. The interior of the sun is so hot that the atoms are completely ionized and the media is a plasma. The light in the hotter interior layers continues to scatter and thus is thermalized with the ions and electrons. At the surface, the temperature has cooled enough that neutral atoms can form and in this layer the medium is transparent and these photons emerge. This same scenario applies to the universe as a whole. Looking out is looking back. The early universe was not only dense but very hot. The evidence for this today is the high entropy of the universe. There are  $10^5 \frac{\text{photons}}{\text{nucleon}}$ . Thus, we see only the surface at about 300,000 years because before that the photons and matter are thermal at temperature too high to have atomic forms. After that time, they are sufficiently soft and the matter can combine to neutral atoms and they no longer scatter and these are the ones that come into our detectors. Obviously, this is when the universe has cooled adiabatically to a temperature about the same as that of the surface of the sun, about 3000 °K. If this were the end of the story we would still have a bright sky between the galaxies. In addition, as the light from the surface of last scatter at age 300,000 years travel to the detectors, the universe has expanded and the light has been red shifted to longer wavelengths. The light thus appears to be from a body that has cooled adiabatically to a very low temperature and is identified as the 3<sup>0</sup>K background radiation, see Section 18.2.8 on page 401. Thus in the modern interpretation, there is no paradox. We do not see the glow of an infinity of stars. Instead, we see the a low temperature glow that is the remnant of the young universe. To the order expected this glow is isotropic, see Section 18.2.9 on page 401.

#### 18.2.4 Hubble Expansion

Originally realized observationally, the Hubble Law was the statement that remote galaxies are moving away from us and that the recession velocity is directly proportional to the distance of the galaxy from us,

$$\vec{v}_{gal} = H\vec{R}. \quad (18.2)$$

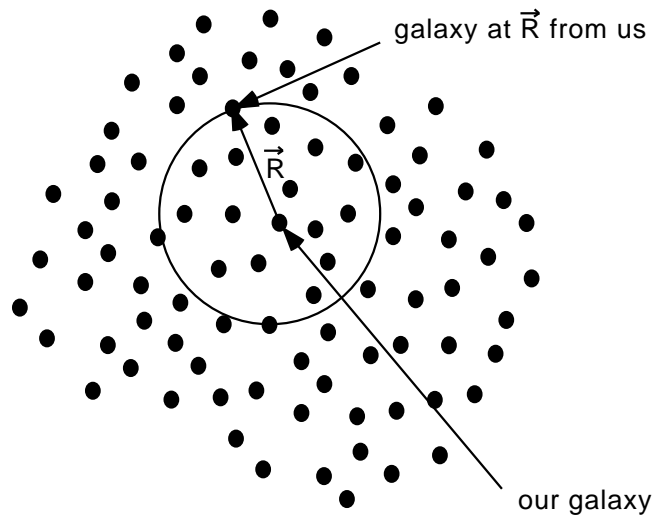


Figure 18.2: **Hubble's Expansion** Hubble's observation that the galaxies are systematically moving away from us. As observed from our galaxy, a galaxy at a relative position  $\vec{R}$  from us, has a velocity  $\vec{v} = H\vec{R}$ . Galaxies at the same distance  $R$  have the same speed. The velocity is directed along the relative position away from us. The figure is drawn with our galaxy near the center. You should realize that this is artistic license and does not imply that our galaxy is located in a special part of the universe, see Section 18.2.2 on page 376.

This simple relationship is the basis of all modern cosmology. The original observations were not very compelling, see Figure 18.3 on page 382. Not only were there few data points but the uncertainty in measuring the distances were rather large. In addition, for nearby galaxies, there may be local motion that distorts the effect. Only on really large distances does the cosmic expansion dominate the velocity. In order to verify this relationship, you need separate measures of distance and velocity. The velocity is actually the easier to measure because of the Doppler shift. The distances are more difficult. Using standard stars, such as variable stars which have a very small range of luminosities, the luminosity can be used to gauge the distance. In fact, now a days, the Hubble Law is now one of the best measures of distance for objects far enough away that the local relative motion is negligible when compared to the cosmic motion. The Hubble plot is a convincing affirmation of the Law, See Figure 18.3.

It took a great deal of faith to base a theory of the universe on this data

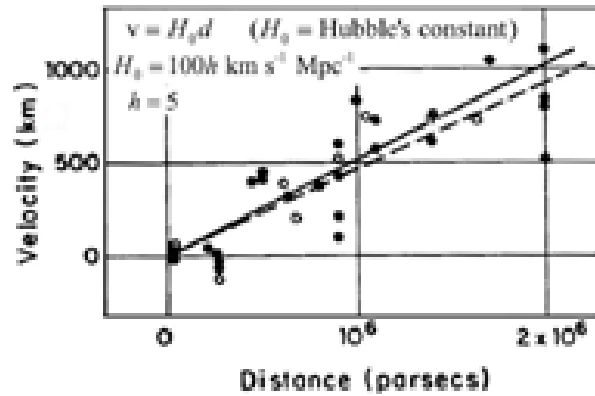


Figure 18.3: **Hubble's Original Plot** The data on the expansion of the universe as presented by Hubble in his original paper in 1929. Note the error in the units on the left axis. Subsequent observations have confirmed the conjecture about the expansion of the universe, see Figure ?? on page ??.

but subsequent analysis has confirmed the conjecture. Note that the Hubble Constant,  $H$ , by its definition, is independent of relative displacement,  $\vec{R}$ , but it can be a function of time. At the time of the laws original formulation, the Hubble Constant was thought to be constant in time but it should be clear that, in any dynamical model of the universe, it will depend on time and in all current models of the universe it does. Of course, if it is a function of time, it is changing at a rate set by the time scale of the universe and, thus, very slowly varying to us. We will thus follow the accepted convention and call it the Hubble Constant despite our anticipation that it varies with time.

A very important fact to note about the Hubble Law is that the Hubble Constant is a scalar; all galaxies at the same distance have the same magnitude of velocity and the direction of that velocity is along the line of sight from us to the galaxy in question, see Figure 18.2 on page 381.

This is a strong conformation of the isotropy of the universe. The only directed quantity that enters the law is the relative displacement. There is no directionality coming from the properties of the universe. The universe is acting like a Pascal fluid, see Section 16.5.1 on page 355. We will take advantage of this fact in preparing simple models of the expansion, see

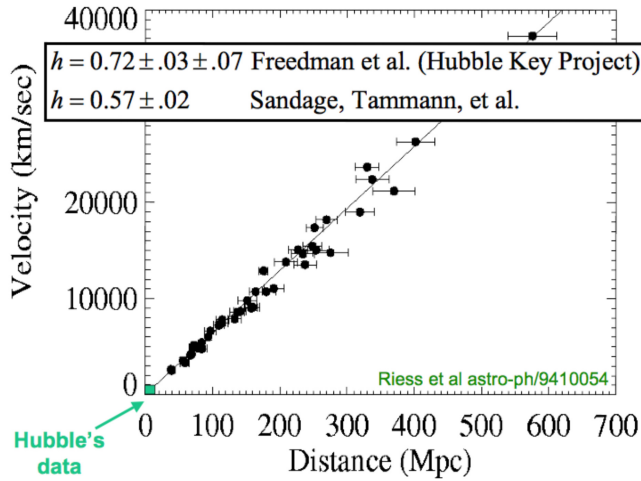


Figure 18.4: **Current Data on Recession of Galaxies** Data from 1994 on the recession of the galaxies. The range of distance is more the seventy times that which was available to Hubble and confirms a linear recession law.

Section 18.2.7 on page 400.

In addition, the Hubble Law is an important confirmation of the homogeneity of the universe. First, we have to make a small technical correction to the form of the law. As discovered by Hubble, the law applied only to reasonably nearby galaxies and the measured velocities were small compared to the speed of light. From our discussion of the hangle, see Section 12.5 on page 289, it should be clear that the hangle is a better measure of relative velocity,  $\chi \equiv \tanh^{-1} \left( \frac{v}{c} \right) \approx \frac{v}{c}$ . It is the additive measure of the Lorentz transformations. Thus,

$$c\vec{\chi} = H\vec{r} \quad (18.3)$$

where  $\vec{\chi}$  has magnitude  $\chi \equiv \tanh^{-1} \left( \frac{v}{c} \right)$  and direction along  $\frac{\vec{v}}{v}$ . The real motivation for this change will be clear as our argument develops. For simplicity of argument, consider a one dimensional universe and an expansion pattern that is arbitrary,  $v(R)$  or better  $\chi(R)$ . Consider the universe and Hubble relationship that would be obtained from a galaxy that is displaced from us by an amount  $d$ . Call this galaxy's relationship  $\chi_d(R_d)$  where  $R_d$  is the distances as measured from that galaxy. We can obtain this pattern with a shift from our galaxy to the new observer galaxy by translating from

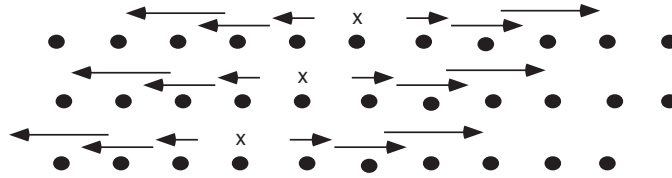


Figure 18.5: **Hubble Law in a Homogeneous Universe** Galaxies distributed uniformly in space are seen in three different reference frames. For example the top universe is viewed from our galaxy. The second down is viewed from our nearest neighbor galaxy. To arrive at the this view, the system is translated and a Lorentz boost is used to bring that galaxy to rest. Our galaxy is now receding. The universe from this new view must be the same as the universe as we view it. Similarly, the bottom line is the universe as viewed from the next galaxy over from our neighbor and looks exactly like our universe.

our location to galaxy  $d$  away and doing a Lorentz transformation of  $\chi(d)$  to come to rest on that galaxy. In order to have the same physics and thus the same Hubble Law at the original and the new location,  $\chi_d(R_d) = \chi(R_d)$ . The translation yields a hangle field  $\chi_{1_d}(R_d) = \chi(R_d - d)$ . The Lorentz transformation yields  $\chi_d(R_d) = \chi(R_d - d) + \chi(d) = \chi(R_d)$  or using  $R_d = R + d$ ,  $\chi(R) + \chi(d) = \chi(R + d)$ . Seeking solutions of the form  $a_n R^n$ , the only solutions are  $n = 0, 1$ . The  $n = 0$  case is eliminated with the requirement  $\chi(0) = 0$ . Thus we see that homogeneity, translation symmetry, and Lorentz invariance implies the Hubble relationship that distance which is additive for the translation and  $\chi$  which is additive for the Lorentz transformation have to be in a linear relationship, Equation 18.3 on page 383.

Probably the most significant feature of the Hubble Law is that it provides for the idea of a finite age for the universe. Reverse all the velocities of expansion and the universe compresses into a dense system, ultimately infinite density in a finite time, see Section 18.2.6 on page 385. This is a particularly simple model for the dynamics of the universe but not overly unrealistic. The fact that the Hubble Law provides us with a dimensionful constant that characterizes the universe is enough to infer a finite lifetime for the universe. The dimension of  $H$  is  $t^{-1}$ . Thus,  $\frac{1}{H}$  is a time. As stated earlier, Section 18.2.1 on page 373, if gravitation is the determining force for the large scale structure of the universe and the universe is homogeneous so that there is only a mass density, there is no time scale in the theory. Thus  $H^{-1}$  provides that time scale and, in any reasonable model of the universe,

the age of the universe will be of the order of  $H^{-1}$ . In fact, when people quote an age for the universe, they are reporting on the latest estimate of  $H^{-1}$ .  $H^{-1}$  is difficult to measure precisely but observations are settling around a number of the order of  $10^{10}$  years. This is a very satisfying number in the sense that we have not been able to find anything significantly older, see Section 18.2.5 on page 385.

### 18.2.5 The Age of the Universe

### 18.2.6 Models of Expanding Universes\*

#### Milne Universe

The simplest model of the universe that incorporates the expansion is called the Milne Universe. Fill the forward light cone of an origin event of Minkowski space with galaxies at all relative velocities. At first, we will discuss a (1, 1) universe but the generalization to (1, 3) is direct. The space-time is represented in Figure 18.6 on page 386. The set of trajectories are  $x_v = vt_v$  or

$$\begin{aligned} t_v &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \tau = \tau \cosh \chi \\ x_v &= \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \tau = c\tau \sinh \chi \end{aligned} \quad (18.4)$$

where the parameter  $v$  or even better the hangle or rapidity,  $\chi$ , see Section 12.5 on page 289, designate the respective trajectories. In Equation 18.4 on page 385,  $-\infty < \chi < \infty$  and  $0 < \tau < \infty$ . In a comoving coordinate system these trajectories would carry the galaxies. In other words, a coordinate system labeled by  $(\chi, \tau)$  would look like an expanding Hubble universe.

The surfaces, in this case curves, of constant  $\tau$  are space-like and infinite,

$$ds^2 = (dx^2 - c^2 dt^2) |_{\tau=\tau_0} = \left\{ \left( \frac{\partial x}{\partial \chi} \right)^2 - \left( c^2 \frac{\partial t}{\partial \chi} \right)^2 \right\} d\chi^2 = c^2 \tau_0^2 d\chi^2, \quad (18.5)$$

or for fixed  $\tau$ ,  $s = c\tau\chi$ .

In this universe, the Hubble expansion is obvious,  $Hs = v$ . This follows simply from  $\frac{s}{c\tau_0} = \chi$ , see Figure 18.7 on page 386. For low velocities,  $\chi = \tanh^{-1} \left( \frac{v}{c} \right) \approx \frac{v}{c}$  which implies  $H = \frac{1}{\tau_0}$  or in general for a universe at age  $\tau$ ,  $H = \frac{1}{\tau}$ . Thus, the Hubble constant is not a constant in time.

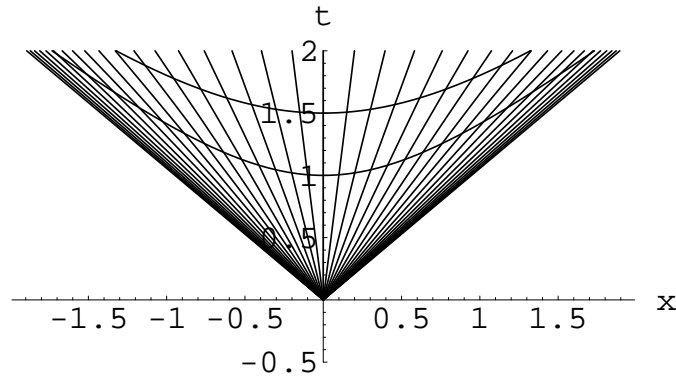


Figure 18.6: **A Milne Universe** The simplest relativistic model of an expanding universe in  $(1, 1)$ . Galaxies carry local coordinates and these are distributed homogeneously in space and hangle. Two space-like surfaces are also shown.

Actually, the Hubble law is better expressed in terms of the hangle since it is the additive measure for the Lorentz transformations, see Section 12.5 on page 289,

$$Hs = c\chi. \quad (18.6)$$

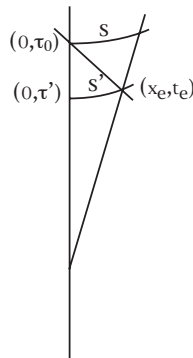


Figure 18.7: **Galaxy Observation in a Milne Universe** A galaxy at a distance  $s$  has a hangle  $\chi = \frac{s}{c\tau_0}$ . This galaxy is viewed at a distance  $s'$  and at the same hangle,  $\chi = \frac{s'}{c\tau'}$ .

In this form, it is also important to note that the Hubble law is then the only velocity or hangle law consistent with the homogeneity of the universe, see Section 18.2.4 on page 380, and Figure 18.5 on page 384. In a  $(1, 3)$  we

will also see that it is an important indicator of isotropy.

Because for large  $v$ ,  $\frac{v}{c}$  near 1, which implies large  $s$  which in turn implies an earlier time and place, see Figure 18.7 on page 386, the original Hubble form must be corrected even further. Calling our galaxy the trajectory at  $v = 0 \Rightarrow \chi = 0$  and thus the coordinatizing galaxy, and  $\tau_0$  as now, a galaxy currently a distance  $s$  from us is seen to be at a distance  $s'$  at a universe age of  $\tau'$  as shown in Figure 18.7 on page 386. The relationship between the times  $\tau_0$  and  $\tau'$  is the Doppler shift times discussed in Section 11.3.3 on page 260 and thus is given by Equation ?? on page ?? where  $t_a$ , the receiving time, is  $\tau_0$  and  $\tau_e$ , the emitting time, is  $\tau'$  or

$$\tau_0 = \tau' \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

or in terms of the hangle,  $\chi = \frac{s}{c\tau_0}$ ,

$$\tau_0 = \tau' \sqrt{\frac{1 + \tanh(\chi)}{1 - \tanh(\chi)}}. \quad (18.7)$$

Since the hangle to the current position and time and the observed position and time is the same,

$$\chi = \frac{s}{c\tau_0} = \frac{s'}{c\tau'}. \quad (18.8)$$

Plugging all this together

$$Hs' = \chi \sqrt{\frac{1 - \tanh(\chi)}{1 + \tanh(\chi)}}. \quad (18.9)$$

Today, the observed objects are at such a distance that the recession velocities are very close to  $c$  and thus these corrections need to be included.

Obviously, at any time  $\tau_0$ , the observer galaxy will have all the other galaxies trajectories in its past at some time. A more interesting and relevant question is how much of any earlier universe is in the past of the observer universe now. Again, Figure 18.7 on page 386 and Equations 18.8 on page 387 and 18.7 on page 387 are relevant. At any time  $\tau_0$ , the observer galaxy sees the galaxy labeled  $\chi$  at age  $\tau'$  and at distance  $s'$ . The relevant point is that although throughout this discussion, we have used the phrase galaxy at distance  $s'$  and hangle  $\chi$  what was really intended has been a patch of an evolving universe. Although a certain patch may at some time contain

a galaxy, because galaxy formation takes time, this patch of universe as seen now may have only cosmic dust and not be seen in the sense that the patch is not luminous and thus conversely is transparent to light from further earlier patches of the universe viewed in that same direction. What we see is the first glowing object in any direction. Fortunately, there do not seem to be many visibly glowing elements now, see Section [refSec:OlbersParadox](#), and in some directions we have a clear view of earlier universes.

To discuss this problem, we need to expand our discussion of Milne universes to (1,3) spaces so that we can have different directions. In a (1,1) space you could never see past the nearest galaxy. The definition of a (1,3) universe is similar to the (1,1) case with the extra condition that the the space portion of the space-time is homogeneous and isotropic. In other words, the observer universe is at the center of a sphere and all directions are identical. Along any direction, a  $(\theta, \phi)$ , from the observer universe a galaxy or patch of universe is moving away at a hangle  $\chi$  and  $\chi = Hs$  where  $H$  is the Hubble constant and  $s$  is the distance from the observer universe now. As we saw earlier, this universe is not only isotropic but symmetric under translations along the direction  $(\theta, \phi)$ . A galaxy at a distance  $s$  and direction  $(\theta, \phi)$  is made the observer universe by boosting by  $\chi = \frac{Hs}{c}$ . In this case, the coordinates are transformed by

$$\begin{aligned} t &= \tau \cosh\left(\frac{Hs}{c}\right) \\ r &= c\tau \sinh\left(\frac{Hs}{c}\right), \end{aligned} \quad (18.10)$$

or

$$\begin{aligned} dt &= \cosh\left(\frac{Hs}{c}\right) d\tau + \frac{H\tau}{c} \sinh\left(\frac{Hs}{c}\right) ds \\ dr &= \sinh\left(\frac{Hs}{c}\right) c d\tau + H\tau \cosh\left(\frac{Hs}{c}\right) ds. \end{aligned} \quad (18.11)$$

Thus usual underlying Minkowski metric is transformed into

$$\begin{aligned} c^2 dT^2 &\equiv c^2 t^2 - dr^2 - r^2 (d\theta^2 + \sin^2(\theta) d\phi^2) \\ &= c^2 d\tau^2 - H^2 \tau^2 ds^2 \\ &\quad - c^2 \tau^2 \sinh^2\left(\frac{Hs}{c}\right) \{d\theta^2 + \sin^2(\theta) d\phi^2\} \end{aligned} \quad (18.12)$$

With this coordinate system in hand, we can seriously discuss a simple (1,3) universe. In this context, Figure 18.6 on page 386 is still relevant but

with the  $x$  label on the horizontal axis interpreted as the  $r$  coordinate in Equations 18.10 on page 388. The figure obviously includes the  $r$  coordinate in the direction antipodal to  $(\theta, \phi)$ . *shinola*

More relevantly, as described in Section 18.2.3 on page 379, if the universe is expanding and has any entropy, there should be a time when the entire universe is at a very high temperature and dense so that it glows like the surface of a star, the surface of last scatter of the early universe. This universe is one that has a temperature of about 3000 °C. The observant reader may wonder why if these patches of universe are the co-moving matter and energy located at that place we do not also include the effects of thermal pressure on the patches. In the Milne universe the patches are inertial by definition and, if we insist that the trajectories are co-moving with the matter and radiation and we have a normal thermal system of matter and radiation, we have an internal inconsistency in the model. This is not the only inconsistency of the Milne model, it has no gravity, but the model is an useful start and subsequently we will add features to complete our model of the universe. For now it is useful to identify the relationship between the galaxy now,  $\tau_0$ , at distance  $s$  and when it

### An Expanding Newtonian Cosmology

For the analysis of this section, we will use non-relativistic physics. This can always work in the sense that we keep the distances and thus the relative velocities small. In addition, we are considering the current epoch of the universe and the energy density is dominated by matter. This analysis will also allow us to separate the effects of ordinary Newtonian Gravitation from the geometric effects of General Relativity. In Section 18.2.6, we will examine a simple General Relativistic Cosmology.

As a measure of the expansion, we will keep track of the distance to some ring of galaxies which are currently at a distance  $R(t)$ . Following the Hubble Law this ring of galaxies is moving away from us at a speed  $\dot{R}(t) = HR$  and these galaxies are gravitationally bound by the sphere of matter contained inside that radius. In the sense of a General Relativistic analysis, we are tracking the expansion in a commoving coordinate system attached to the galaxies at  $R(t)$ .

Let us examine the dynamics of the expansion as they arise naturally from the Newtonian force law on a galaxy of mass  $m$  at  $R(t)$ . Define the quantity  $M_{inside}$  as the mass inside the sphere of radius  $R(t)$ . Obviously,

the mass density  $\rho$  is

$$\rho = \frac{3}{4\pi R(t)^3} M_{inside}, \quad (18.13)$$

Newton's force law yields

$$\begin{aligned} m a(t) &= m \ddot{R}(t) = -\frac{GmM_{inside}}{R(t)^2} \\ &= -\frac{Gm\rho \frac{4\pi R(t)^3}{3}}{R(t)^2} \end{aligned} \quad (18.14)$$

or

$$\ddot{R}(t) = -\frac{4\pi}{3} \rho G R(t). \quad (18.15)$$

Note that this equation is negative definite, gravitation is universally attractive. It is clearly the case that in a homogeneous universe  $\rho$  can only depend on time. Thus the first integral of this equation with respect to time and usually identified with the energy has to be handled with care. Replacing  $\rho$  with  $M_{inside}$ , Equation 18.13, and assuming that  $M_{inside}$  is a constant in time, we get

$$\frac{d}{dt} \left( \dot{R}(t)^2 \right) = 2M_{inside}G \left( \frac{d}{dt} \left( \frac{1}{R(t)} \right) \right).$$

Integrating and putting  $\rho$  back

$$\dot{R}(t)^2 = \frac{8\pi}{3} \rho G R(t)^2 + K \quad (18.16)$$

where  $K$  is a constant of integration. This result has a simple interpretation in terms of the energy of the galaxies at the edge of a sphere of radius  $R$ , see Figure 18.2. For a galaxy of mass  $m$ , the potential energy is

$$PE = -\frac{GmM_{inside}}{R} = -\frac{4\pi R^2 \rho m G}{3}. \quad (18.17)$$

The kinetic energy for a galaxy of mass  $m$  at this distance is

$$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{R}(t)^2. \quad (18.18)$$

Thus, the total energy of the galaxy at  $R(t)$  is

$$\begin{aligned} E(R) &= KE + PE \\ &= \frac{1}{2}m\dot{R}(t)^2 - \frac{4\pi R^2 \rho m G}{3} \\ &= \frac{1}{2}mK. \end{aligned}$$

Using the definition of the Hubble constant,  $v = HR$ , the

$$KE = \frac{1}{2}mH^2R^2. \quad (18.19)$$

Thus the total energy of galaxies at the distance  $R$  is

$$\begin{aligned} E(R) &= KE + PE \\ &= mR^2\left\{\frac{1}{2}H^2 - \frac{4}{3}\pi\rho G\right\} \\ &= \frac{mR^2H^2}{2}\left\{1 - \frac{8\pi\rho G}{3H^2}\right\} \end{aligned} \quad (18.20)$$

Note that, because of the homogeneity assumption,  $H$  and  $\rho$  are independent of position. This energy is positive or negative at all  $R$  and is the same sign no matter what the value of  $R$ . Thus the sign of this energy is a measure that is universal in the universe. We will find later, Equation 18.26, that, if the energy is negative, the galaxies will stop expanding and later start to fall back. Thus if  $E$  is positive, the galaxies will continue to expand indefinitely. Thus, there is a critical mass density of the universe that denotes the boundary between continued indefinite expansion and slow down and ultimate collapse.

Using the dimensional content of  $H$  and  $G$ , we can define a mass/energy density

$$\rho_{crit} \equiv 3\frac{H^2}{8\pi G}. \quad (18.21)$$

Since  $H$  is universal this is the critical density everywhere as expected on the basis of homogeneity. Also since  $H = \frac{dR}{R}$  where as stated above  $R$  is a commoving coordinate, if there is acceleration in the commoving coordinate,  $H$  and thus the critical density changes with time.

The energy of a galaxy currently at distance  $R_N$  from us is

$$E(R_N) = \frac{mH_N^2R_N^2}{2}\left(1 - \frac{\rho_N}{\rho_{crit N}}\right), \quad (18.22)$$

where the subscripts  $N$  indicate that we are using the current value.

Defining

$$\Omega \equiv \frac{\rho}{\rho_{crit}}, \quad (18.23)$$

where both densities are taken at the same time, this energy is

$$E(R_N) = \frac{mH_N^2R_N^2}{2}(1 - \Omega_N). \quad (18.24)$$

The criteria for the positivity of the expansion energy of the universe in the current epoch is simply whether or not  $\Omega_N > 1$ .

### Equation for evolution of the scale factor

The energy expression, Equation 18.20, can be used to calculate the evolution of  $R(t)$ . It is interesting to note that we have been calculating a Newtonian Cosmology. There is no field theory of gravity with finite propagation effects or general or special relativistic corrections. This turns out to be okay because of the judicious choice of the comoving coordinate system. Later we will look at the General Relativistic approach, see Section ?? and compare that approach with this one. The advantage of this Newtonian analysis besides its conceptual simplicity is the references to our usual intuition of dynamics. The three things that we are doing that would not have been appropriate to a true Newtonian cosmology is identifying the evolutionary nature of the universe associated with the cosmological expansion, identifying the space time with the galactic expansion, and using as the source of gravity the mass/energy. In addition, none of the current analysis treats issues of geometry of space let alone space time.

Using Equation 18.20, the energy per unit mass of a galaxy on the shell at  $R_N$  is

$$\frac{E(R_N)}{m} = \frac{1}{2}H_N^2 R_N^2 - G\frac{4\pi}{3}\rho_N R_N^2 \quad (18.25)$$

In the same notation, the energy for the galaxies in the same shell at a latter time is

$$\begin{aligned} \frac{E(R(t))}{m} &= \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - G\frac{4\pi}{3}\rho_N \frac{R_N^3}{R(t)} \\ &= \frac{1}{2}\left(\frac{dR}{dt}\right)^2 - \frac{H_N^2 R_N^2}{2}\Omega_N \frac{R_N}{R(t)} \end{aligned} \quad (18.26)$$

where the mass/energy contained within the shell,  $M_{inside R_N} \equiv \frac{4\pi}{3}\rho_0 R_N^3$ , has been conserved.

Equation 18.26 has the same dependence as the one for an object of unit mass being projected to a height,  $h = R(t)$ , on a body of mass  $M_{inside R_N}$ . Thus if we require conservation of energy for comoving elements for all time,  $E(R(t)) = E(R_N)$ , then, if  $E(R_N)$  is positive,  $\frac{dR}{dt}$  will increase indefinitely and, in a sense, escape the massive body. If  $E(R(t))$  is negative, the projected body would have slowed and eventually turn around and start to fall back.

For instance, setting  $E(R(t)) = E(R_N)$ , or, better said the energy of expansion, Equation 18.24, we find that, if  $\Omega_N$  is greater than one, the greatest distance that a galaxy, which is currently at distance  $R_N$ , will be

from us is

$$\begin{aligned} R_{max} &= \frac{8\pi G\rho_N R_N}{3H_N^2(\Omega_N - 1)} \\ &= R_N \frac{\Omega_N}{\Omega_N - 1}. \end{aligned} \quad (18.27)$$

Similarly, if  $\Omega_N < 1$ ,  $\frac{dR}{dt} > 0$  for all time.

The expansion energy can be used to find the general expression for  $\frac{dR}{dt}$ ,

$$\left(\frac{dR}{dt}\right)^2 = H_N^2 R_N^2 \left(1 - \Omega_N \left(1 - \frac{R_N}{R(t)}\right)\right). \quad (18.28)$$

Since  $\frac{dR}{dt} > 0$ , the positive root is the appropriate choice.

$$\left(\frac{dR}{dt}\right) = H_N R_N \sqrt{1 - \Omega_N \left(1 - \frac{R_N}{R(t)}\right)}. \quad (18.29)$$

Both for reasons of simplicity and ease of interpretation, it is best to use rescaled variables, the distance in units of  $R_N$ ,  $\alpha \equiv \frac{R(t)}{R_N}$  and times in units of  $H_N^{-1}$ ,  $\tau \equiv H_N t$ , Equation 18.29, takes the particularly simple form

$$\frac{d\alpha}{d\tau} = \sqrt{1 - \Omega_N \left(1 - \frac{1}{\alpha}\right)}. \quad (18.30)$$

$\alpha$  is often called the scale factor of the universe.

Two features of this result are important to note. Firstly, we have a one parameter,  $\Omega_N$ , family of universes. Depending on the value of  $\Omega_N$ , and only on  $\Omega_N$ , the universe will either forever expand or reverse expansion and collapse. If  $\Omega_N > 1$ , the term in the square root is always positive and the system will expand forever. If  $\Omega_N < 1$ , the term with the square root can vanish and the universe will collapse back onto itself. Secondly, the acceleration is easy to compute,

$$\frac{d^2\alpha}{d\tau^2} = -\frac{\Omega_N}{2\alpha^2}. \quad (18.31)$$

There is no surprize in this result. This is Newton's Law of Gravitation applied to the commoving galaxy in these new variables. In fact, the first integral of this expression is the energy of expansion, Equation 18.20. This acceleration is negative definite. Gravity is the only force operating and it is

always attractive. In fact, measurement of a positive acceleration is a special problem for this approach to cosmology. Recent observations indicating the presence of a positive acceleration, [?], present a special problem for this approach. We will see that, in the General Relativistic approach, there is the possibility of positive accelerations but that it will require a form of matter that is not consistent with our current understanding of microscopic physics or an uncomfortable value for the cosmological constant, see Section ??.

In addition, Equation 18.30 is easy to integrate although the closed form solution is not particularly useful. The boundary condition is obviously  $\alpha(\tau = \text{now}) = 1$ . Choosing the origin of time such that  $\tau = \text{now} = 1$ , we can plot the evolution of the scale factor for times earlier than now, see Figure 18.8 and in Figure 18.9 for longer times for three values of the  $\Omega_N$ ;  $\Omega_N = 0.5, \Omega_N = 1$ , and  $\Omega_N = 1.5$ . In Figure 18.8, The universe starts from the time that the scale factor vanishes. It can be seen from Figure 18.8 that the current age of the universe is not strongly dependent on  $\Omega_N$  and is the order of the inverse Hubble constant as expected.

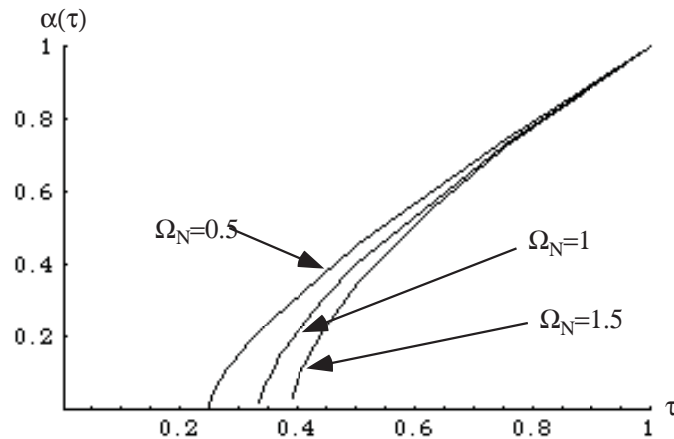


Figure 18.8: **Evolution of the scale factor for early times** The evolution of the scale factor depends only on the mass/energy in the universe. Three cases for the mass/energy density are shown:  $\Omega_N = 0.5$  which is an ever expanding universe,  $\Omega_N = 0.5$  which is at the transition between collapsing and ever expanding, and  $\Omega_N = 1.5$  which is collapsing universe

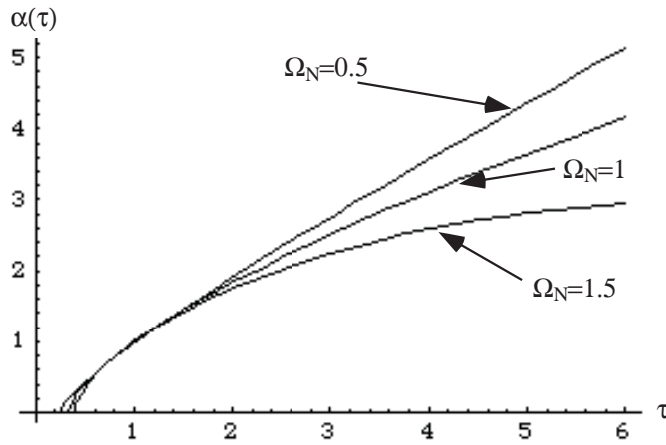


Figure 18.9: **Long time dependence of the scale factor** The evolution of the scale factor depends only on the mass/energy in the universe.

### Evolution of Density

Using the fact that the mass/energy in any comoving shell is conserved,  $M_{inside R(t)} = M_{inside R_N}$ , the density scaling law becomes

$$\rho = \frac{\rho_N}{\alpha^3}. \quad (18.32)$$

Putting this expression into Equation 18.31, the acceleration of the scale factor becomes

$$\frac{d^2\alpha}{d\tau^2} = -\frac{4\pi}{3}\rho\alpha\frac{G}{H_N^2}. \quad (18.33)$$

This result shows the Newtonian gravitational basis for the acceleration of the scale factor, it is not as useful as it may appear since we need to find the evolution of the density to integrate it. From the density scaling law, Equation 18.32,

$$\frac{d\rho}{d\tau} = -3\frac{\rho}{\alpha}\frac{d\alpha}{d\tau} \quad (18.34)$$

$$= -3\rho\frac{H}{H_N}. \quad (18.35)$$

Again this expression is not as useful as it seems. We require the solution for  $H(\tau)$  in order to integrate it.

Similarly, the evolution of the density in terms of  $\alpha$  follows from the scaling law, Equation 18.32 and Equation 18.34 as

$$\begin{aligned}\frac{d\rho}{d\tau} &= -3\frac{\rho_N}{\alpha^4}\frac{d\alpha}{d\tau} \\ &= -3\frac{\rho_N}{\alpha^4}\sqrt{1-\Omega_N}\left(1-\frac{1}{\alpha}\right).\end{aligned}\quad (18.36)$$

Given the solution of Equation 18.30, this equation can be integrated to give the evolution of the density.

### Evolution of H

Given the acceleration of the scale factor, Equation 18.31, it is straight forward to get the equation for the evolution of H

$$\begin{aligned}\frac{d}{d\tau}\left(\frac{H}{H_N}\right) &= \frac{d}{d\tau}\left(\frac{\frac{d\alpha}{d\tau}}{\alpha}\right) \\ &= \frac{\frac{d^2\alpha}{d\tau^2}}{\alpha} - \frac{\left(\frac{d\alpha}{d\tau}\right)^2}{\alpha^2} \\ &= -\frac{\Omega_N}{2\alpha^3} - \left(\frac{H}{H_N}\right)^2 \\ &= -\frac{1}{\alpha^2}\left(1-\Omega_N + \frac{3\Omega_N}{2\alpha}\right).\end{aligned}\quad (18.37)$$

which is manifestly negative definite as expected.

This model contains all of the large scale features of what is termed the “Big Bang” cosmology. There are features of this model that have not been dealt with such as the nature of the mass/energy in the universe. These will be dealt with later when microphysics has been included. Suffice at this point to say that the matter considered is ordinary matter that obeys all the usual rules of macroscopic and microscopic matter physics such as thermodynamics and our latest discoveries of elementary particle physics. These matters will all be discussed in Section 19.2.1 on page 404. In addition, there has been no discussion of the space/time geometry. This will require the use of General Relativity which is dealt with in Section 18.2.7 on page 400.

One property of the mass/energy that is clearly important is the amount.  $\Omega_N$  is the only parameter that labels our models of the universe and thus determines whether the universe will expand forever or will eventually fall back on itself and collapse, see Section 18.2.6

### Friedman Robertson Walker Space-Time

A Friedman Robertson Walker space-time is homogeneous and isotropic in space and obeys the Einstein equation in a (1, 3) space. We have experience with homogeneous isotropic two spaces in three space. Embedding the three generic examples, we have the two sphere constrained by  $x^2 + y^2 + z^2 = R^2$ , the flat plane with the constraint  $z = 0$ , and the hyperboloid of revolution with the constraint  $x^2 + y^2 - c^2t^2 = R^2$  in a (1, 2) Minkowski space.

These can be unified into a simple single form for the metric. Starting with the two sphere,  $x^2 + y^2 + z^2 = R^2 \Rightarrow 2xdx + 2ydy + 2zdz = 0 \Rightarrow dz = -\frac{xdx+ydy}{\sqrt{R^2-x^2-y^2}}$ ,

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= dx^2 + dy^2 + \left( \frac{xdx + ydy}{\sqrt{R^2 - x^2 - y^2}} \right)^2, \end{aligned}$$

where  $R$  is the constant radius of the two sphere.

Going to polar coordinates in the  $xy$  plane,  $x = r \cos \theta$ , and  $y = r \sin \theta$ ,

$$dl^2 = \frac{R^2 dr^2}{R^2 - r^2} + r^2 d\theta^2, \quad (18.38)$$

or defining a dimensionless radius,  $r_0 \equiv \frac{r}{R}$ ,

$$dl^2 = R^2 \left\{ \frac{dr_0^2}{1 - r_0^2} + r_0^2 d\theta^2 \right\}. \quad (18.39)$$

The homogeneous isotropic negative curvature two surface is the hyperboloid of revolution  $x^2 + y^2 - c^2t^2 = -R^2$  embedded in a (1, 2) Minkowski space-time. This is the surface at fixed proper time  $c\tau = R$ . Following the same pattern as before,  $cdt = \frac{xdx+ydy}{R^2+x^2+y^2}$  or

$$dl^2 = dx^2 + dy^2 - \left( \frac{xdx + ydy}{R^2 + x^2 + y^2} \right)^2.$$

Again going to polar coordinates,

$$dl^2 = \frac{R^2 dr^2}{R^2 + r^2} + r^2 d\theta^2, \quad (18.40)$$

Changing to the dimensionless radius,

$$dl^2 = R^2 \left\{ \frac{dr_0^2}{1 + r_0^2} + r_0^2 d\theta^2 \right\}. \quad (18.41)$$

The flat case is obtained by taking the limit  $R \rightarrow \infty$  in either Equation 18.38 or 18.44 or

$$dl^2 = dr^2 + r^2 d\theta^2. \quad (18.42)$$

Using the fact that the curvature

$$\kappa = \begin{cases} \frac{1}{R^2} & : \text{ for the positively curved case} \\ 0 & : \text{ for the flat case} \\ -\frac{1}{R^2} & : \text{ for the negatively curved case} \end{cases}, \quad (18.43)$$

all three cases are given by

$$dl^2 = \frac{dr^2}{1 - \kappa r^2} + r^2 d\theta^2. \quad (18.44)$$

or

$$dl^2 = R^2 \left\{ \frac{dr_0^2}{1 - k r_0^2} + r_0^2 d\theta^2 \right\}, \quad (18.45)$$

where

$$k = \begin{cases} 1 & : \text{ for the positively curved case} \\ 0 & : \text{ for the flat case} \\ -1 & : \text{ for the negatively curved case} \end{cases} \quad (18.46)$$

is the sign of the curvature.

The extension to an isotropic homogeneous three space in a  $(1, 3)$  space is found by replacing the angular measure from the unit one sphere, the circle, with the usual measure on the unit two sphere. Thus our metric is

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - k r^2} + r^2 d\theta^2 \right\}, \quad (18.47)$$

where  $R(t)$  is called the scale factor of the universe. For this metric, the non-zero curvature components are for the Ricci tensor

$$\begin{aligned} R_{00} &= -3 \frac{\ddot{R}}{R} \\ R_{ij} &= - \left[ \frac{\ddot{R}}{R} + 2 \frac{\dot{R}^2}{R^2} + 2 \frac{k}{R^2} \right] g_{ij} \end{aligned} \quad (18.48)$$

and the curvature scalar is

$$R = -6 \left[ \frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} \right] \quad (18.49)$$

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### Missing Mass

As can be expected, it is very difficult to measure the mass/energy density of the universe. There are several reasons for this. We are not in a region of the universe that is typical. Our planet is in a solar system about a star that is in a galaxy that is a part of a local cluster of galaxies. The star that we orbit is at least a second generation star and thus the matter that is around us is not cosmic in origin. Most significantly, until very recently, the only observable tool was the light from or absorbed by the matter. In fact, all that you can directly observe is the luminous matter. You have to infer the mass from the nature of the light.

### Luminous Matter

The standard procedure is to look at the glow of standard objects whose mass can be inferred from other properties of the object. Models of stellar structure provide a tight relationship between the glow of stars and their mass. Galaxies are made of stars and thus we can infer the mass of the glowing material of the galaxies. Thus a ratio of luminosity to mass and assumed proportionality can be established for the mass associated with all the luminous objects observed in the universe and from this a density of matter. In all cases, for the systems in consideration, the mass dominates the mass/energy density. Of course, there could be cool dark objects and often you will hear arguments for their contribution to the mass density of the universe. The occurrence of these kinds of things at a rate sufficient to contribute significantly to the mass density provides theoretical astronomers with lots of speculative freedom and opportunities to publish. It should also be clear that this estimate is at best correct to within a factor of two. The current best estimate is that the mass associated with luminous matter is

$$\Omega_{Nlum} \approx 0.01 \quad (18.50)$$

or less.

### Gravitational Mass

Besides using the luminous matter, we can infer mass from its gravitational effects. Assuming that the stars in galaxies are gravitationally bound and If you look at the speed of stars then you can estimate the mass that is the source of the gravity that is binding them.

### Figure of rotation curves

The mass required to provide dynamic equilibrium is approximately 10 times the luminous mass. This increases the critical density to

$$\Omega_{N_{grav}} \approx 0.1 \quad (18.51)$$

In addition, the galaxies are clustered. We are in group called the local cluster. If you assume that these clusters are not accidental combinations but are also gravitationally bound, there is dark mass between the galaxies. Adding in this mass increases the critical density to

$$\Omega_{N_{clus}} \approx 0.2 \quad (18.52)$$

Einstein had a theoretical prejudice for a universe with  $\Omega_N > 1$ . We have not yet discussed the space time structure of the universe, see Section 18.2.7, but in the same way that the values of  $\Omega_N$  determines the collapse or expansion of the universe, it determines the nature of the geometry. This should be no surprize since a collapse would imply a finite timelike geodesic. In a fully relativistic treatment, a finite timelike world line implies finite space-like geodesics and, thus, a finite universe. In this case, there is no need for boundary conditions on the universe at its start. Thus, there was a reason to feel that there should be more matter in the universe than that which was observed by the these two methods. This became known as the “missing mass” problem. More recently, there has been a theoretical prejudice for the case  $\Omega_N = 1$ . This is driven by the need for an inflationary phase at the start of the universe, see Section 23.2. Regardless, there was a strong desire to find more matter than could be seen, luminous, or felt, gravitational. The problem now is that positive accelerations have now been observed and the best description of the large scale structure of the universe, the “Standard Model” , Section 18.2.11 requires dark matter and dark energy. Neither of these seem to be consistent with our current understanding of the nature of matter as developed in microphysics.

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### 18.2.7 The Space Time Structure

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Before elaborating further on the difficulties with a simple expansion model of the universe, we will redo the analysis of the above section, Section 18.2.6, using the tools of general relativity still restricting ourselves to a simple picture of the nature of the matter in the universe. This will enable us to understand the geometry of the universe and to better understand the role of the dark energy.

Using the arguments of homogeneity and isotropy you can show that the general form of the metric is

$$c^2 d\tau^2 = c^2 dt^2 - R^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 d^2\Omega \right\} \quad (18.53)$$

where  $R(t)$  is a function of time and is determined by Einstein's equation if you know the energy and momentum densities.  $R(t)$  is called the scale factor of the universe.  $k$  is a constant that takes on the values 1, 0, or -1.

Using this metric you can get all the curvatures. The three three space curvatures are equal and are  $\frac{k}{R^2}$ . Thus the three space is positively curved for  $k = 1$ . It is flat if  $k = 0$  and negatively curved for  $k = -1$ . For  $k = 1$  the geodesics are all finite in length and thus have finite volume. The other two spaces have infinite geodesics and thus infinite volumes. We can thus identify the three cases that we have here with the values of the critical density that we had above.  $\Omega_N > 1$  is the closed positively curved universe.  $\Omega_N = 1$  is the case of the flat space and  $\Omega_N < 1$  is the negatively curved universe. These last two cases have infinitely long straight lines.

Whether or not the universe is finite or infinite is determined by the mass density of the universe. It is clear that the value of  $\Omega_N$  is an important parameter.

### 18.2.8 Black Body Background

### 18.2.9 Problems with the Expanding Universe

### 18.2.10 The Cosmological Constant

### 18.2.11 The Standard Model of the Universe

